An explicit expression of the basic relative invariants of homogeneous convex cones

Hideto Nakashima

Kyushu university
(JSPS Research Fellow)

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Background

Theorem (Vinberg 1963)

Homogeneous convex domains	$\Leftrightarrow$	Clans
Homogeneous convex cones	$\Leftrightarrow$	Clans with unit

$\Omega \subset V$: homogeneous cone
$\Delta_1(x), \ldots, \Delta_r(x)$: basic relative invariants of Ω

$$\Omega = \{x \in V;\ \Delta_1(x) > 0, \ldots, \Delta_r(x) > 0\}.$$

$R_x y := y \triangle x$: right multiplication operator

$$\mathrm{Det}\, R_x = \Delta_1(x)^{n_1} \cdots \Delta_r(x)^{n_r} \quad (n_j \geq 1).$$

Clans

V : finite-dimensional real vector space

$\triangle$: bilinear product in V

Definition

$(V, \triangle)$ is a clan $\Leftrightarrow$ the following three conditions are satisfied:

(C1) $[L_x, L_y] = L_{x \triangle y - y \triangle x}$, (left symmetric algebra)

(C2) $\exists s \in V^*$ s.t. $s(x \triangle y)$ is an inner product, (compactness)

(C3) L_x has only real eigenvalues. (normality)

($L_x y := x \triangle y$: left multiplication operator)

In general, clans are $\begin{cases} \text{non-associative,} \\ \text{non-commutative,} \\ \text{no unit element.} \end{cases}$

Examples

$\boldsymbol{V} = \mathbf{Herm}(\boldsymbol{r}, \mathbb{K}), \quad (\mathbb{K} = \mathbb{R}, \mathbb{C},$ or $\mathbb{H})$.

- $\boldsymbol{x} \triangle \boldsymbol{y} := \underline{\boldsymbol{x}}\, \boldsymbol{y} + \boldsymbol{y}(\underline{\boldsymbol{x}})^* \ (\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V})$.

$$\underline{\boldsymbol{x}} := \begin{pmatrix} \frac{1}{2}x_{11} & 0 & \cdots & 0 \\ x_{21} & \frac{1}{2}x_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ x_{r1} & x_{r2} & \cdots & \frac{1}{2}x_{rr} \end{pmatrix}.$$

Normal decomposition

- V : clan with unit element e_0,
- $c_1, \ldots, c_r$: complete system of orthogonal primitive idempotents
  $(c_i \triangle c_j = \delta_{ij} c_i,\ c_1 + \cdots + c_r = e_0)$
- Normal decomposition: $V = \bigoplus_{1 \le j \le k \le r} V_{kj}$, where

$$\begin{cases} V_{jj} = \mathbb{R} c_j \quad (j = 1, \ldots, r), \\ V_{kj} = \left\{ x \in V;\ L_{c_i} x = \frac{1}{2}(\delta_{ij} + \delta_{ik}) x,\ R_{c_i} x = \delta_{ij} x \right\}. \end{cases}$$

In the case of $V = \mathrm{Sym}(r, \mathbb{R})$,

- $c_j = E_{jj}$,
- $V_{kj} = \mathbb{R}(E_{kj} + E_{jk})$.

Basic relative invariants

- $\mathfrak{h} := \{L_x;\ x \in V\}$ (split solvable Lie algebra).
- $H := \exp \mathfrak{h}$.
- $\Omega := H \cdot e_0 \Rightarrow$homogeneous cone.
  In particular, $H \curvearrowright \Omega$: simply transitively.

Definition.

1. $f\colon \Omega \to \mathbb{R}$: relatively H-invariant
   $\Leftrightarrow \exists \chi\colon H \to \mathbb{R}$: 1-dim. representation s.t. $f(hx) = \chi(h) f(x)$.
2. $\Delta_j(x)$: relatively H-invariant irreducible polynomials
   $(j = 1, \ldots, r)$
   $\Rightarrow$the basic relative invariants

Remark. **(Ishi 2001, Ishi–Nomura 2008)**
$\forall p(x)$: relatively H-invariant polynomial

$$\Rightarrow p(x) = (\mathrm{const})\Delta_1(x)^{m_1} \cdots \Delta_r(x)^{m_r}\ (m_1, \ldots, m_r \in \mathbb{Z}_{\geq 0}).$$

If $p(x) = \mathrm{Det}\, R_x$, then we have $m_k \geq 1$ $(k = 1, \ldots, r)$.

Dual clan

Definition

$(V, \triangledown)$: the dual clan of V

$$\langle x \triangledown y \,|\, z \rangle = \langle y \,|\, x \vartriangle z \rangle \quad (x, y, z \in V).$$

$$\begin{array}{lccc} \text{homogeneous cone} & \Omega & \longleftrightarrow & \Omega^* \\ & \updownarrow & \text{dual} & \updownarrow \\ \text{clan} & (V, \vartriangle) & \longleftrightarrow & (V, \triangledown) \end{array}$$

- Relation between $\vartriangle$ and $\triangledown$:

$$x \vartriangle y + x \triangledown y = y \vartriangle x + y \triangledown x.$$

Examples

$V = \mathbf{Herm}(r, \mathbb{K})$, $(\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H})$.

- $x \triangle y := \underline{x}\, y + y(\underline{x})^* \ (x, y \in V)$.

$$\underline{x} := \begin{pmatrix} \frac{1}{2}x_{11} & 0 & \cdots & 0 \\ x_{21} & \frac{1}{2}x_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ x_{r1} & x_{r2} & \cdots & \frac{1}{2}x_{rr} \end{pmatrix}.$$

- Corresponding cone: $\Omega = \{x \in V;$ positive definite$\}$.
- basic relative invariants: $\Delta_k(x) = \det^{(k)}(x)$.
- $\mathrm{Det}\, R_x = \Delta_1(x)^d \cdots \Delta_{r-1}(x)^d \Delta_r(x)$ $(d = \dim \mathbb{K})$.
- Dual clan product:

$$x \bigtriangledown y = (\underline{x})^* y + y\, \underline{x} \quad (x, y \in V).$$

Representations of clans

$\boldsymbol{E}$: a real Euclidean vector space with $\langle\cdot|\cdot\rangle_{\boldsymbol{E}}$

Definition

Let $\varphi\colon \boldsymbol{V} \to \mathcal{L}(\boldsymbol{E}) = \{\text{Linear maps on } \boldsymbol{E}\}$.
$(\varphi, \boldsymbol{E})$: a selfadjoint representation of the dual clan $(\boldsymbol{V}, \triangledown)$:

- $\varphi(\boldsymbol{x})^* = \varphi(\boldsymbol{x})$ and $\varphi(\boldsymbol{e_0}) = \mathbf{id}_{\boldsymbol{E}}$,
- $\varphi(\boldsymbol{x} \triangledown \boldsymbol{y}) = \overline{\varphi}(\boldsymbol{x})\varphi(\boldsymbol{y}) + \varphi(\boldsymbol{y})\underline{\varphi}(\boldsymbol{x})$,

where $\underline{\varphi}(\boldsymbol{x})$ (resp. $\overline{\varphi}(\boldsymbol{x})$) is lower (resp. upper) triangular part of φ.

Definition. $\boldsymbol{Q}\colon \boldsymbol{E} \times \boldsymbol{E} \to \boldsymbol{V}$: bilinear map associated with φ:

$$\langle \boldsymbol{Q}(\boldsymbol{\xi}, \boldsymbol{\eta}) \,|\, \boldsymbol{x} \rangle = \langle \varphi(\boldsymbol{x})\boldsymbol{\xi} \,|\, \boldsymbol{\eta} \rangle_{\boldsymbol{E}} \quad (\boldsymbol{\xi}, \boldsymbol{\eta} \in \boldsymbol{E},\ \boldsymbol{x} \in \boldsymbol{V}).$$

$\boldsymbol{Q}[\boldsymbol{\xi}] := \boldsymbol{Q}(\boldsymbol{\xi}, \boldsymbol{\xi})$ and $\boldsymbol{Q}[\boldsymbol{E}] := \{\boldsymbol{Q}[\boldsymbol{\xi}];\ \boldsymbol{\xi} \in \boldsymbol{E}\}$.

Clan defined by representation

Theorem

$V_E^0 := \mathbb{R}u \oplus E \oplus V$ with

$$X \triangle Y = (\lambda\mu)u + (\mu\xi + \tfrac{1}{2}\lambda\eta + \underline{\varphi}(x)\eta) + (Q(\xi,\eta) + x \triangle y).$$
$$(X = \lambda u + \xi + x,\ Y = \mu u + \eta + y).$$

$\Rightarrow (V_E^0, \triangle)$ is a clan of rank $r+1$ with unit element.

pr. Use the following properties of φ and Q:

- $x \triangle Q(\xi,\eta) = Q(\underline{\varphi}(x)\xi,\eta) + Q(\xi,\underline{\varphi}(x)\eta)$,
- Q is Ω-positive, i.e. $Q[\xi] \in \overline{\Omega}$ (for non-zero $\xi \in E$),
- $\underline{\varphi}(x \triangle y - y \triangle x) = [\underline{\varphi}(x), \underline{\varphi}(y)]$.

Calculate $\mathrm{Det}\, R_X^0$ and express by using $\mathrm{Det}\, R_x$

Right multiplication operators

$\boldsymbol{R^0}$:Right multiplication operator of $\boldsymbol{V_E^0}$

$$\text{Then we have} \quad R^0_{\lambda u+\xi+x} = \begin{pmatrix} \lambda & 0 & 0 \\ \frac{1}{2}\xi & \lambda \mathrm{id}_E & R^0_\xi \\ 0 & R^0_\xi & R_x \end{pmatrix},$$

where $\boldsymbol{R}$ is right multiplication operator of $\boldsymbol{V}$.

Proposition

$$\mathrm{Det}\, R^0_{\lambda u+\xi+x} = \lambda^{1+\dim E-\dim V}\, \mathrm{Det}\, R_{\lambda x-\frac{1}{2}Q[\xi]}$$

Right multiplication operators

The basic relative invariants of V_E^0 are exhausted by

$$\begin{cases} \lambda, \\ \text{irreducible factors of } \Delta_j(\lambda x - \frac{1}{2}Q[\xi]) \quad (j = 1, \ldots, r), \end{cases}$$

where $\Delta_j(x)$ are the basic relative invariants of V.

Theorem

$P_j(X)$: the basic relative invariants of V_E^0.
There exist integers $\alpha_j \geq 0$ s.t.

$$P_j(X) = \begin{cases} \lambda & (j = 0), \\ \lambda^{-\alpha_j}\Delta_j(\lambda x - \frac{1}{2}Q[\xi]) & (j = 1, \ldots, r). \end{cases}$$

In order to determine α_j, we introduce

1. multiplier matrix,
2. ε-representation.

Multiplier matrix

- H : split solvable Lie group
  $\Rightarrow \exists h_{jj} \in \mathbb{R}_{>0}$ and $\exists h_{kj} \in V_{kj}$ s.t.

$$h = (\exp T_{11})(\exp L_1)(\exp T_{22}) \cdots (\exp L_{r-1})(\exp T_{rr}).$$

$(T_{jj} = (2 \log h_{jj}) L_{c_j}$ and $L_j = L_{h_{j+1,j}} + \cdots + L_{h_{rj}})$

- f : relatively H-invariant ($\exists \chi$ s.t. $f(hx) = \chi(h) f(x)$)

$$\Rightarrow \exists \tau_j \in \mathbb{R} \text{ s.t. } f(he_0) = (h_{11})^{2\tau_1} \ldots (h_{rr})^{2\tau_r} f(e_0).$$

$\underline{\tau} := (\tau_1, \ldots, \tau_r)$: multiplier of f.

Multiplier matrix

Definition

$\underline{\sigma}_j = (\sigma_{j1}, \ldots, \sigma_{jr})$: multiplier of Δ_j

$$(\Delta_j(he_0) = (h_{11})^{2\sigma_{j1}} \cdots (h_{rr})^{2\sigma_{jr}})$$

σ: the multiplier matrix $\Leftrightarrow \sigma = \begin{pmatrix} \underline{\sigma}_1 \\ \vdots \\ \underline{\sigma}_r \end{pmatrix} = (\sigma_{jk})_{1 \le j,k \le r}$

Remark. (**Ishi 2001**)

- σ is (lower) triangular,
- $\sigma_{jj} = 1$.

$$\text{i.e.} \quad \sigma = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \sigma_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \sigma_{r1} & \sigma_{r2} & \cdots & 1 \end{pmatrix}.$$

ε-representation

(φ, E): representation of $(V, \triangledown)$
$\varepsilon = {}^t(\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r$, $\quad c_\varepsilon := \varepsilon_1 c_1 + \cdots + \varepsilon_r c_r$.

Definition

$$(\varphi, E): \ \varepsilon\text{-representation} \Leftrightarrow Q[E] = \overline{H \cdot c_\varepsilon}.$$

Remark. Put $\mathcal{O}_\varepsilon := H \cdot c_\varepsilon$. Then one has

$$\overline{\Omega} = \bigsqcup_{\varepsilon \in \{0,1\}^r} \mathcal{O}_\varepsilon \qquad \text{(Ishi 2000).}$$

Proposition

(φ, E): any representation
$\exists 1\ \varepsilon = \varepsilon(\varphi) \in \{0,1\}^r$ s.t. φ is an ε-representation.

ε-representation

Calculation of $\varepsilon(\varphi)$

Put $d_{kj} := \dim V_{kj}$.

$$l^{(1)} := {}^t(\dim E_1, \ldots, \dim E_r) \qquad (E_j := \varphi(c_j)E),$$

$$l^{(k)} := \begin{cases} l^{(k-1)} - {}^t(0, \ldots, 0, d_{k,k-1}, \ldots, d_{r,k-1}) & (l^{(k-1)}_{k-1} > 0), \\ l^{(k-1)} & \text{(otherwise)}. \end{cases}$$

Then $\varepsilon(\varphi) = {}^t(\varepsilon_1, \ldots, \varepsilon_r)$ is defined by

$$\varepsilon_k = \begin{cases} 1 & (\text{if } l^{(k)}_k > 0), \\ 0 & \text{(otherwise)}. \end{cases}$$

Determination of α_j

Theorem

$$\begin{cases} V & : \text{ clan of rank } r \\ (\varphi, E) : & \varepsilon\text{-representation} \end{cases} \longrightarrow (V_E^0, \triangle) : \text{ clan of rank } r+1$$

$P_j(X)$: basic relative invariants of V_E^0 $(j = 0, 1, \ldots, r)$.

$$P_j(\lambda u + \xi + x) = \lambda^{-\alpha_j} \Delta_j(\lambda x - \tfrac{1}{2} Q[\xi]).$$

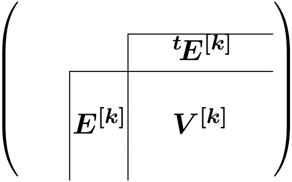
Let σ be the multiplier matrix of V. Then one has

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = \sigma \begin{pmatrix} 1 - \varepsilon_1 \\ \vdots \\ 1 - \varepsilon_r \end{pmatrix}$$

multiplier matrix of V_E^0: $\sigma^0 = \begin{pmatrix} 1 & 0 \\ \sigma\varepsilon & \sigma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varepsilon & I_r \end{pmatrix}$.

Multiplier matrix (2)

- Put $\begin{cases} E^{[k]} := \bigoplus_{m>k} V_{mk}, \\ V^{[k]} := \bigoplus_{k<l\le k\le r} V_{ml}. \end{cases}$



- $\mathcal{R}^{[k]}(x)\xi := \xi \bigtriangledown x \ (x \in V^{[k]},\ \xi \in E^{[k]})$.
  $\Rightarrow$representation of $(V^{[k]}, \bigtriangledown)$.
- Put $\varepsilon^{[k]} := \varepsilon(\mathcal{R}^{[k]})$ and $\mathcal{E}_k := \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varepsilon^{[k]} & I_{r-k} \end{pmatrix}$.

Theorem

$$\sigma_V = \mathcal{E}_{r-1}\mathcal{E}_{r-2}\cdots\mathcal{E}_1,$$

i.e. the multiplier matrix is determined only by the dimensions of V_{kj}.

Vinberg's polynomials

- Definition of Vinberg's polynomials $D_j(x)$ are as follows:
- $\|x\|^2 := \langle x \,|\, x \rangle$.
- Define $x^{(j)} = \sum_{k=j}^{r} x_{kk}^{(j)} c_k + \sum_{m>k\geq j} x_{mk}^{(j)} \in V^{[j-1]}$ by

$$
\begin{aligned}
x^{(1)} &:= x, \\
x_{kk}^{(j+1)} &:= x_{jj}^{(j)} x_{kk}^{(j)} - \frac{1}{2s_0(c_k)} \|x_{kj}^{(j)}\|^2 \quad (j < k \leq r), \\
x_{mk}^{(j+1)} &:= x_{jj}^{(j)} x_{mk}^{(j)} - x_{mj}^{(j)} \triangle x_{kj}^{(j)} \quad (j < k < m \leq r).
\end{aligned}
$$

- Then

$$D_j(x) := x_{jj}^{(j)} \in \mathbb{R}.$$

Explicit expression

Main theorem

Let $\sigma = (\sigma_{ij})_{1 \le i \le j \le r}$ be the multiplier matrix of V. Then one has

$$\Delta_1(x) = D_1(x), \quad \Delta_j(x) = \frac{D_j(x)}{\prod_{i<j} D_i(x)^{\tau_{ji}}},$$

where $\tau_{ji} = -\sigma_{ji} + \sigma_{j,i+1} + \cdots + \sigma_{jj} \in \mathbb{Z}_{\ge 0}$.

To determine $\Delta_j(x)$:

Divide $D_j(x)$ by $D_i(x)$ until not-divisible

$\downarrow$

Divide $D_j(x)$ by $D_i(x)$ τ_{ji}-times.

Thank you for your attention!